SM3 8.2 Exponential & Log Problems

Mr. Sinister has infected 6 of the X-men with his vampiric virus on October 1st. Each of those X-men infects the people around them at a rate modeled by the equation $y = ae^{0.012t}$, where t is measured in days.

1. Is this a growth or decay model?

The argument of the exponent will be positive. Therefore, $e^{()}$ will be larger than 1. This is a growth model.

2. What is the rate of growth or decay?

The coefficient of the variable, t, in the exponent is the rate. The rate of growth is 0.012 per day.

3. What is the initial number of infected people?

Mr. Sinister infected 6 of the X-men. The initial number of infected people is $a = 6$.

4. What is the time, t, in the calculation of people infected in one month's time?

The number of days from Oct 1st to Nov 1st is 31. The time, $t = 31$, is the number of days in a month since October 1st.

5. What is the formula used to solve the problem?

As $a = 6$ and $t = 31$, $y = (6)e^{0.012(31)}$

6. How many people will be infected in one month's time?

$y = 6e^{0.372} \approx 8.7$ or about 8.7 people.

Beast has discovered a cure for Mr. Sinister's virus. It is an electro-magnetic pulse that travels from his location outward. Beast sets off the pulse on November 1st. The pulse instantly cures Beast, who was infected, and also cures the other people who have been effected at a rate modeled by $y = ae^{0.052t}$, where t is measured in days. The pulse also halts any further spread of the infection the moment it fires.

7. Is this a growth or decay model?

The argument of the exponent will be positive. Therefore, $e^{()}$ will be larger than 1. This is a growth model for people being cured. It should be noted that Mr. Sinister might consider this to be a decay model as it will decay the number of people he has infected!

8. What is the rate of growth or decay?

The coefficient of the variable, t, in the exponent is the rate. The rate of growth is 0.052 per day.

9. What is the number of cured people when Beast begins the electro-magnetic pulse?

Mr. Sinister infected 6 of the X-men, and that number grew to about 8.7 people by Nov 1st, when Beast fired up the pulse. Beast was instantly cured, so the initial number of cured people is 1.

10. How many people could Beast cure using the pulse during the month of November?

The number of days from Nov 1st to Dec 1st is $t = 30$. $y = 1e^{1.56} \approx 4.76$ people will be cured.

11. If Beast begins the pulse on Nov 1st, how long until the infection be completely cured?

As $a = 1$ and $y = 8.7$, $8.7 = (1)e^{0.052t}$ is the equation we're trying to solve for t.

$8.7 =$	$1e^{0.052t}$	Given
$8.7 =$	$e^{0.052t}$	No need for coefficient of 1
$\ln 8.7 =$	$0.052t$	ln( ) both sides
$\dfrac{\ln 8.7}{0.052} =$	t	Divide by 0.052

It will take $\dfrac{\ln 8.7}{0.052} \approx 41.6$ days to cure the infection completely.

13. Tiger Industries bought a computer for $2500. It is expected to depreciate at a rate of 20% per year. What will the value of the computer be in 2 years?

$$a = 2500, r = 0.2, t = 2, y = ?$$

$y =$	$a(1-r)^t$	Discontinuous decay
$y =$	$2500(1-0.2)^2$	Substitution
$y =$	$2500(0.8)^2$	Subtraction
$y =$	$2500(0.64)$	Exponents
$y =$	1600	Multiplication

The value of the computer will be $1600 in 2 years.

14. The Martins bought a condominium for $85000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years?

$$a = 85000, r = 0.05, t = 5, y = ?$$

$y =$	$a(1+r)^t$	Discontinuous growth
$y =$	$85000(1+0.05)^5$	Substitution
$y =$	$85000(1.05)^5$	Subtraction
$y \approx$	$85000(1.276)$	Exponents
$y \approx$	108483.93	Multiplication

The value of the condo will be about $108483.93 in 5 years.

Bacteria usually reproduce by a process known as binary fission. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

15. Find the constant *r* for this type of bacteria under ideal conditions.

$$a = 1, y = 2, r = ?, t = 20$$

$y =$	$a(1+r)^t$	Discontinuous growth
$2 =$	$(1+r)^{20}$	Substitution
$\pm\sqrt[20]{2} =$	$1+r$	Take the 20th root of both sides
$-1 \pm \sqrt[20]{2} =$	r	Subtraction
$-1 + \sqrt[20]{2} =$	r	Remove extraneous solution

The constant, r, for this type of bacteria is $r = -1 + \sqrt[20]{2} \approx 0.035$ per minute.

16. Write the equation for modeling the exponential growth of this bacterium.

As $k = 0.035$, $y \approx a(1 + 0.035)^t$ will model the growth of the bacterium.

The annual Gross Domestic Product of a country is the value of all of the goods and services produced in the country during a year. During the period 1995-2009, the Gross Domestic Product of the United State grew about 3.2% per year, measured in 2006 dollars. In 1995, the GDP was $5717 Billion.

17. Assuming this rate of growth continues, what will the GDP of the United States be in 2020?

$$a = 5717 \text{ (billion)}, y = ?, r = 0.032, t = 25$$

$y =$	$a(1+r)^t$	Discontinuous growth
$y =$	$5717(1+0.032)^{25}$	Substitution
$y =$	$5717(1.032)^{25}$	Addition
$y \approx$	$5717(2.1978)$	Exponents
$y \approx$	12564.95	Multiplication

The GDP of the U.S. in 2020 will be about $12565 billion, which is about $12.565 trillion.

18. In what year will the GDP reach $20 trillion?

$$a = 5717 \text{ (billion)}, y = 20000 \text{ (billion)}, r = 0.032, t = ?$$

$y =$	$a(1+r)^t$	Discontinuous growth
$20000 =$	$5717(1+0.032)^t$	Substitution
$20000 =$	$5717(1.032)^t$	Addition
$\frac{20000}{5717} =$	$(1.032)^t$	Division
$\ln\frac{20000}{5717} =$	$\ln(1.032)^t$	ln( ) both sides
$\ln\frac{20000}{5717} =$	$t\ln 1.032$	Log property
$\frac{\ln\frac{20000}{5717}}{\ln 1.032} =$	t	Division

The U.S. GDP will reach $20000 at $t = \frac{\ln\frac{20000}{5717}}{\ln 1.032} \approx 39.76$ years after 1995, which is during 203

19. In 1928, when the high jump was first introduced as a women's sport at the Olympic Games, the winning women's jump was 62.5 inches, while the winning men's jump was 76.5 inches. Since then, the winning jump for women has increased by 0.38% per year, while the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women's winning high jump be higher than the men's?

Women's variables: $a_w = 62.5, y_w = ?, r_w = 0.0038, t = ?$
Men's variable: $a_m = 76.5, y_m = ?, r_m = 0.003, t = ?$

$y_w =$	y_m	The women's jump will be the same distance as the men's jump
$a_w(1+r_w)^t =$	$a_m(1+r_m)^t$	Set the two equations for distance equal.
$62.5(1+0.0038)^t =$	$76.5(1+0.003)^t$	Substitution
$62.5(1.0038)^t =$	$76.5(1.003)^t$	Addition
$\frac{62.5}{76.5} =$	$\frac{(1.003)^t}{(1.0038)^t}$	Division
$\frac{62.5}{76.5} =$	$\left(\frac{1.003}{1.0038}\right)^t$	Exponent property
$\ln\left(\frac{62.5}{76.5}\right) =$	$\ln\left(\frac{1.003}{1.0038}\right)^t$	ln( ) both sides
$\ln\left(\frac{62.5}{76.5}\right) =$	$t\ln\left(\frac{1.003}{1.0038}\right)$	Log property
$\frac{\ln\left(\frac{62.5}{76.5}\right)}{\ln\left(\frac{1.003}{1.0038}\right)} =$	t	Division, cry a little on the inside because of the impending calculation

The women's high jump will be higher than the men's shortly after $t = \frac{\ln\left(\frac{62.5}{76.5}\right)}{\ln\left(\frac{1.003}{1.0038}\right)} \approx 253.5$ years have passed since 1928, which is during 2181.

20. The Brutus family bought a new house 10 years ago for \$120,000. The house is now worth \$191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation?

$a = 120000, y = 191000, r = ?, t = 10$

$y =$	ae^{rt}	Continuous Growth
$191000 =$	$120000e^{10r}$	Substitution
$\frac{191000}{120000} =$	e^{10r}	Division
$\ln\left(\frac{191000}{120000}\right) =$	$10r$	ln( ) both sides
$\frac{\ln\left(\frac{191000}{120000}\right)}{10} =$	r	Division

The yearly rate of appreciation that had steady growth is $r = \frac{\ln\left(\frac{191000}{120000}\right)}{10} \approx 0.046$.

Compound interest

21. If \$2000 is invested with an annual rate of 6%, compounded continuously, how long will it take the money to double? How much money will you have after 10 years?

$a = 2000, y = 4000, r = 0.06, t = ?$

$y =$	ae^{rt}	Continuous Growth
$4000 =$	$2000e^{0.06t}$	Substitution
$2 =$	$e^{0.06t}$	Division
$\ln(2) =$	$0.06t$	ln( ) both sides
$\frac{\ln(2)}{0.06} =$	t	Division

It will take $t = \frac{\ln(2)}{0.06} \approx 11.55$ years for the money to double.

$a = 2000, y = ?, r = 0.06, t = 10$

$y =$	ae^{rt}	Continuous Growth
$y =$	$2000e^{0.06(10)}$	Substitution
$y =$	$2000e^{0.6}$	Division
$y \approx$	$2000(1.8222)$	ln( ) both sides
$y \approx$	3644.24	Division

You will have about \$3644.24 after 10 years.

22. If \$2000 is invested and after 5 years you have \$2665, what is the interest rate, if it is compounded monthly?

$a = 2000, y = 2665, r = ?, t = 5$

$y =$	$a\left(1 + \frac{r}{n}\right)^{nt}$	Per Unit Growth
$2665 =$	$2000\left(1 + \frac{r}{12}\right)^{12*5}$	Substitution
$\frac{2665}{2000} =$	$\left(1 + \frac{r}{12}\right)^{60}$	Division
$\sqrt[60]{\left(\frac{2665}{2000}\right)} =$	$1 + \frac{r}{12}$	Root both sides
$\sqrt[60]{\left(\frac{2665}{2000}\right)} - 1 =$	$\frac{r}{12}$	Subtraction
$12 * \left(\sqrt[60]{\left(\frac{2665}{2000}\right)} - 1\right) = r$		Multiplication

The interest rate that compounded continuously is $12 * \left(\sqrt[60]{\left(\frac{2665}{2000}\right)} - 1\right) \approx$

23. If $2000 is invested with an annual rate of 5%, compounded daily, how long will it to increase the amount of money by $400?

$$a = 2000, y = 2400, r = \frac{0.05}{365}$$
$$, t = ?$$

$y =$	$a\left(1+\frac{r}{n}\right)^{nt}$	Per Unit Growth
$2400 =$	$2000\left(1+\frac{.05}{365}\right)^{356*t}$	Substitution
$\frac{6}{5} =$	$\left(1+\frac{.05}{365}\right)^{365t}$	Division
$\ln\left(\frac{6}{5}\right) =$	$365t * \ln\left(1+\frac{.05}{356}\right)$	ln( ) both sides
$\frac{\ln\left(\frac{6}{5}\right)}{365 * \ln\left(1+\frac{.05}{356}\right)} =$	t	Division

It will take $t \approx 3.55$ years for the money to increase by $400.

24. Find the time necessary for $1000 to double if it is invested at a rate of $r = 7\%$ compounded continuously?

$$a = 1000, y = 2000, r = 0.07, t = ?$$

$y =$	ae^{rt}	Continuous Growth
$2000 =$	$1000e^{0.07t}$	Substitution
$2 =$	$e^{0.07t}$	Division
$\ln(2) =$	$0.07t$	ln( ) both sides

$$\frac{\ln(2)}{0.07} = \quad t \qquad\qquad \text{Division}$$

If compounded continuously, it will take $t = \frac{\ln(2)}{0.07} \approx 9.902$ years for the money to double.